Introduction to Temporal Point Processes (II)

HUMAN-CENTERED MACHINE LEARNING
http://courses.mpi-sws.org/hcml-ws18/
Temporal Point Processes: Basic building blocks
Poisson process

Intensity of a Poisson process

\[ \lambda^*(t) = \mu \]

Observations:

1. Intensity independent of history
2. Uniformly random occurrence
3. Time interval follows exponential distribution
Fitting a Poisson from (historical) timeline

\[
\lambda^*(t) = \mu
\]

\[
\mu^* = \arg \max_{\mu} 3 \log \mu - \mu T = \frac{3}{T}
\]
Sampling from a Poisson process

We sample using inversion sampling:

\[
\lambda^*(t) = \mu
\]

We would like to sample:

\[
t \sim \mu \exp(-\mu(t - t_3))
\]

We sample using inversion sampling:

\[
F_t(t) = 1 - \exp(-\mu(t - t_3)) \quad \Rightarrow \quad t \sim -\frac{1}{\mu} \log(1 - u) + t_3
\]

\[
P(F_t^{-1}(u) \leq t) = P(u \leq F_t(t)) = F_t(t)
\]
Inhomogeneous Poisson process

Intensity of an inhomogeneous Poisson process

\[ \lambda^*(t) = g(t) \geq 0 \]

Observations:

1. Intensity independent of history
Fitting an inhomogeneous Poisson

Design \( g(t) \) such that max. likelihood is convex (and use CVX)
Nonparametric inhomogeneous Poisson process

Positive combination of (Gaussian) RFB kernels:

\[ \lambda^*(t) = \sum_j \alpha_j k(t - t_j) \]
Sampling from an inhomogeneous Poisson

$\mu = \max g(t)$

**Thinning procedure (similar to rejection sampling):**

1. Sample $t$ from Poisson process with intensity $\mu$
   
   $t \sim -\frac{1}{\mu} \log(1 - u) + t_3$

2. Generate $u_2 \sim \text{Uniform}(0, 1)$

3. Keep the sample if $u_2 \leq \frac{g(t)}{\mu}$
Terminating (or survival) process

Intensity of a terminating (or survival) process

\[ \lambda^*(t) = g^*(t)(1 - N(t)) \geq 0 \]

Observations:

1. Limited number of occurrences
Self-exciting (or Hawkes) process

Intensity of self-exciting (or Hawkes) process:

\[
\lambda^*(t) = \mu + \alpha \sum_{t_i \in \mathcal{H}(t)} \kappa_\omega(t - t_i)
\]

Observations:

1. Clustered (or bursty) occurrence of events
2. Intensity is stochastic and history dependent
Fitting a Hawkes process from a recorded timeline

\[ \lambda_0 = \lambda^*(t_3) \]

\[ \lambda^*(t_1) \lambda^*(t_2) \lambda^*(t_3) \ldots \lambda^*(t_n) \exp \left( -\int_0^T \lambda^*(\tau) \, d\tau \right) \]

\[ \lambda^*(t) = \mu + \alpha \sum_{t_i \in \mathcal{H}(t)} \kappa_{\omega}(t - t_i) \]

The max. likelihood is jointly convex in \( \mu \) and \( \alpha \)

\[ \text{maximize} \quad \sum_{i=1}^{n} \log \lambda^*(t_i) - \int_0^T \lambda^*(\tau) \, d\tau \]

(use CVX!)
Sampling from a Hawkes process

Thinning procedure (similar to rejection sampling):

1. Sample \( t \) from Poisson process with intensity \( \mu_3 \)

\[
U \sim Uniform(0, 1) \\
\log(1 - U) + t_3 \\
\frac{1}{\mu_3}
\]

Inversion sampling

2. Generate \( u_2 \sim Uniform(0, 1) \)

3. Keep the sample if \( u_2 \leq g(t) / \mu_3 \)
Building blocks to represent different dynamic processes:

Poisson processes:
\[ \lambda^*(t) = \lambda \]

Inhomogeneous Poisson processes:
\[ \lambda^*(t) = g(t) \]

Terminating point processes:
\[ \lambda^*(t) = g^*(t)(1 - N(t)) \]

Self-exciting point processes:
\[ \lambda^*(t) = \mu + \alpha \sum_{t_i \in \mathcal{H}(t)} \kappa_{\omega}(t - t_i) \]
Summary

Building blocks to represent different dynamic processes:

Poisson processes:
\[ \lambda^*(t) = \lambda \]

Inhomogeneous Poisson processes:
\[ \lambda (t) = g(t)(1 - N(t)) \]

Terminating point processes:

Self-exciitng point processes:
\[ \lambda^*(t) = \mu + \alpha \sum_{t_i \in \mathcal{H}(t)} \kappa_\omega(t - t_i) \]

We know how to fit them and how to sample from them
Temporal Point Processes: Superposition
Superposition of processes

Sample each intensity + take minimum = Additive intensity

\[ t = \min(\tau, \tau_1, \tau_2, \tau_3) \quad \Rightarrow \quad \lambda^*(t) = \mu + \alpha \sum_{t_i \in \mathcal{H}(t)} \kappa_\omega(t - t_i) \]
Mutually exciting process

Clumped occurrence affected by neighbors

\[ \lambda^*(t) = \mu + \alpha \sum_{t_i \in \mathcal{H}_b(t)} \kappa_\omega(t - t_i) + \beta \sum_{t_i \in \mathcal{H}_c(t)} \kappa_\omega(t - t_i) \]
Mutually exciting terminating process

Clustered occurrence affected by neighbors

\[ \lambda^*(t) = (1 - N(t)) \left( g(t) + \beta \sum_{t_i \in \mathcal{H}_c(t)} \kappa_\omega(t - t_i) \right) \]