Introduction to
Temporal Point Processes (II)

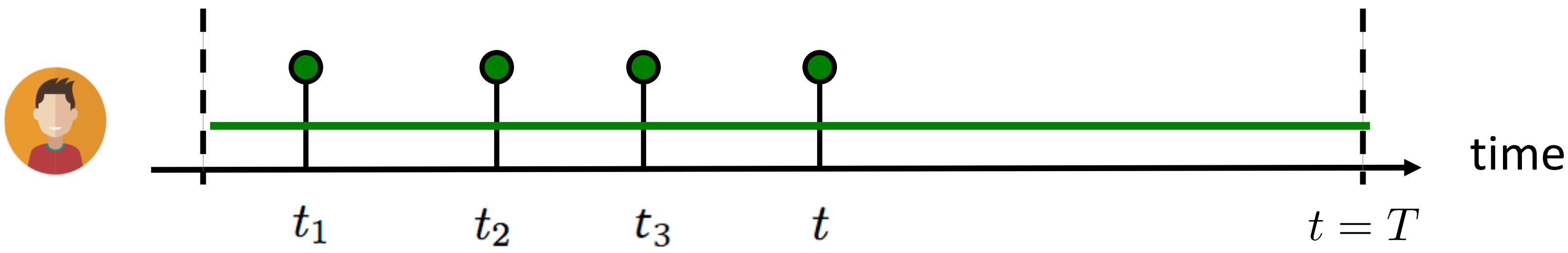
HUMAN-CENTERED MACHINE LEARNING

http://courses.mpi-sws.org/hcml-ws18/

MAX PLANCK INSTITUTE
FOR SOFTWARE SYSTEMS

Temporal Point Processes:
Basic building blocks

Poisson process



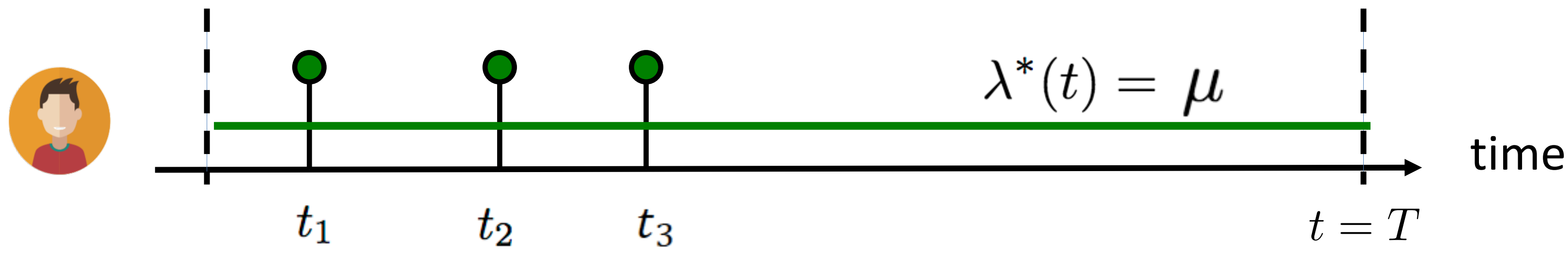
Intensity of a Poisson process

$$\lambda^*(t) = \mu$$

Observations:

1. **Intensity independent of history**
2. **Uniformly random occurrence**
3. **Time interval follows exponential distribution**

Fitting a Poisson from (historical) timeline

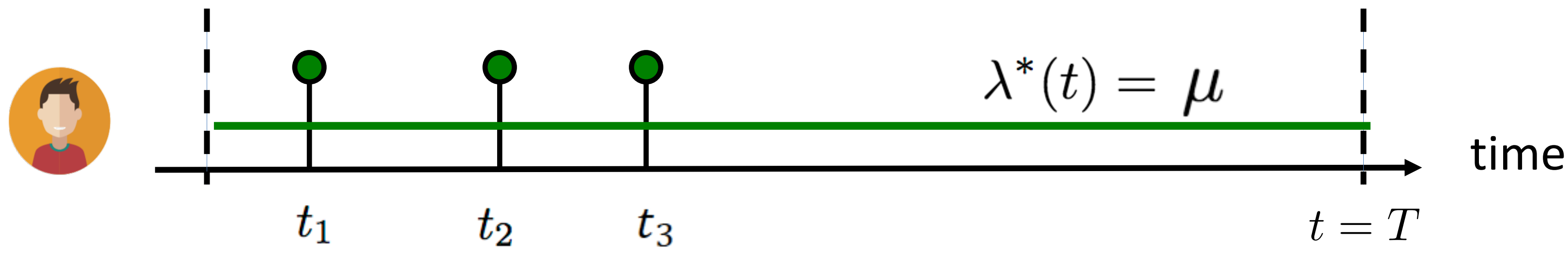


$$\underset{\mu}{\underset{\uparrow}{\lambda^*(t_1)}}\ \underset{\mu}{\underset{\uparrow}{\lambda^*(t_2)}}\ \underset{\mu}{\underset{\uparrow}{\lambda^*(t_3)}}\ \underbrace{\exp\left(-\int_0^T \lambda^*(\tau)\,d\tau\right)}_{\exp(-\mu T)}$$

Maximum likelihood

$$\mu^* = \underset{\mu}{\operatorname{argmax}}\ 3\log\mu - \mu T = \frac{3}{T}$$

Sampling from a Poisson process



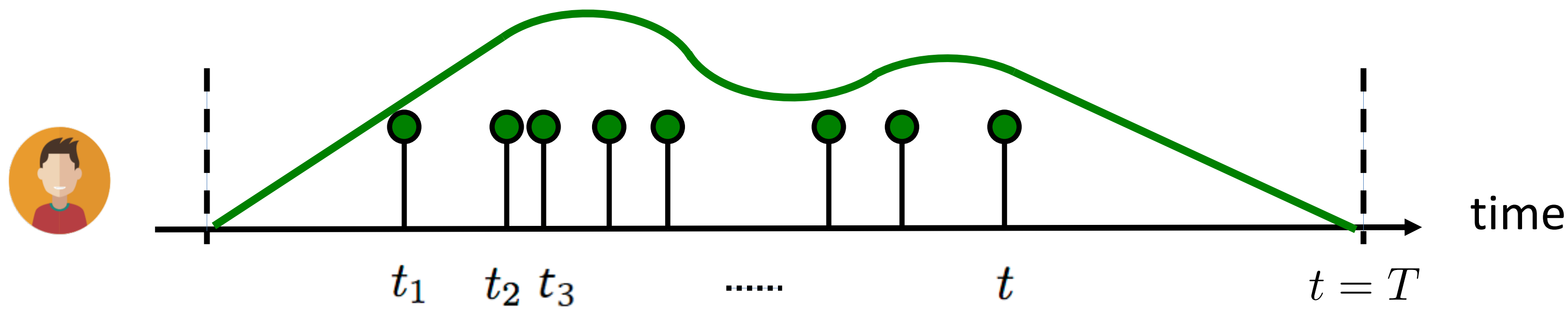
We would like to sample: $t \sim \mu \exp(-\mu(t - t_3))$

We sample using inversion sampling:

$$F_t(t) = 1 - \exp(-\mu(t - t_3)) \Rightarrow t \sim \underbrace{-\frac{1}{\mu}\log(1 - \overset{\overset{Uniform(0,1)}{\downarrow}}{u}) + t_3}_{F_t^{-1}(u)}$$

$$\mathbb{P}(F_t^{-1}(u) \le t) = \mathbb{P}(u \le F_t(t)) \underset{\underset{F_u(u)=u}{\uparrow}}{=} F_t(t)$$

Inhomogeneous Poisson process



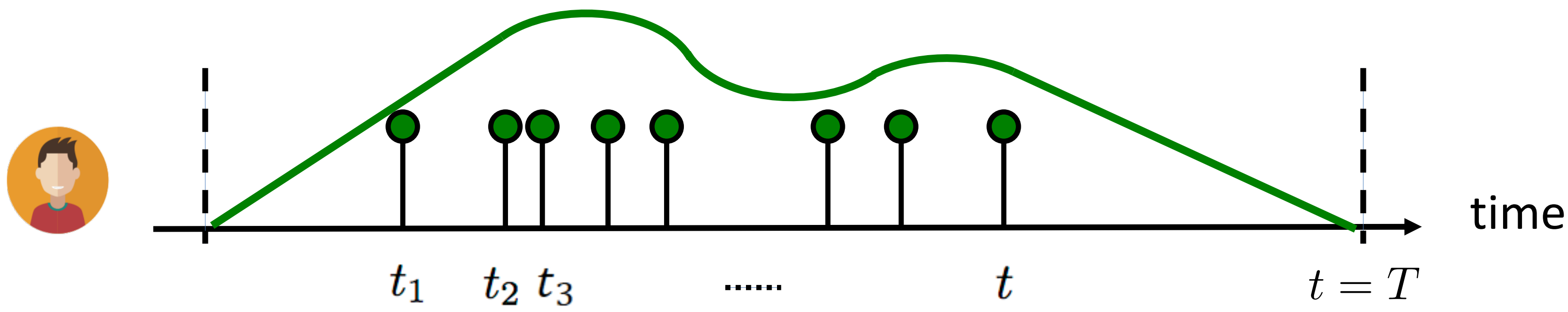
Intensity of an inhomogeneous Poisson process

$$\lambda^*(t) = g(t) \geqslant 0$$

Observations:

1. **Intensity independent of history**

Fitting an inhomogeneous Poisson



$$\lambda^*(t_1)\lambda^*(t_2)\,\lambda^*(t_3) \cdots\cdots \lambda^*(t_n) \;\; \underbrace{\exp\left(-\int_0^T \lambda^*(\tau)\,d\tau\right)}_{\exp\left(-\int_0^T g(\tau)\,d\tau\right)}$$

$$\uparrow \qquad\quad \uparrow \qquad\quad \uparrow \qquad\qquad\qquad \uparrow$$

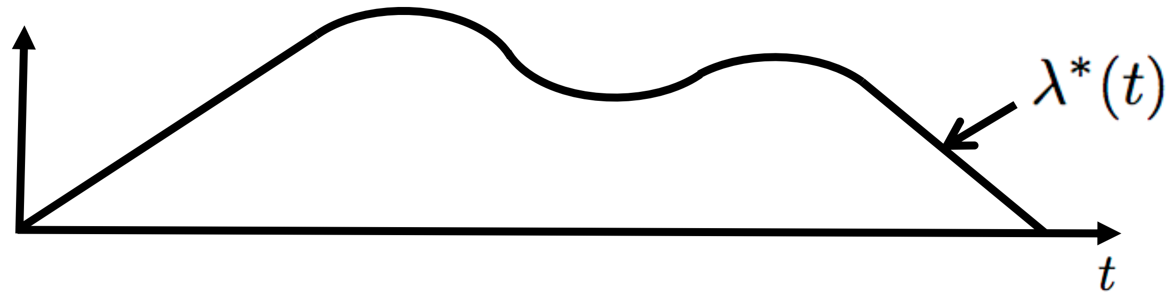
$$g(t_1) \quad g(t_2) \quad g(t_3) \qquad\quad g(t_n)$$

Maximum likelihood

$$\underset{g(t)}{\text{maximize}} \quad \sum_{i=1}^{n} \log g(t_i) - \int_0^T g(\tau)\,d\tau$$

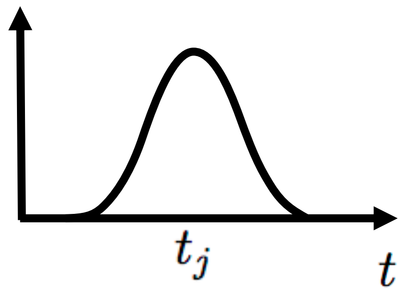
Design $g(t)$ such that max. likelihood is convex (and use CVX)

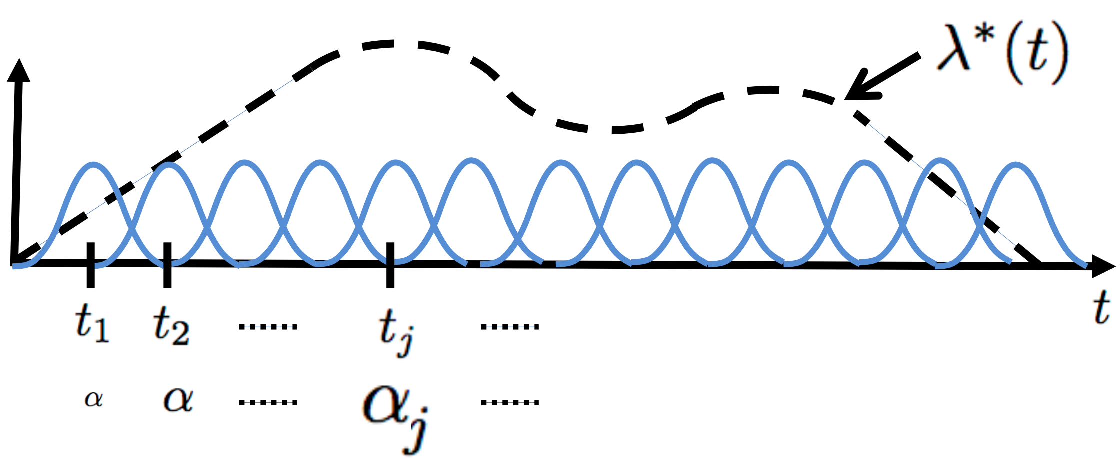
Nonparametric inhomogeneous Poisson process



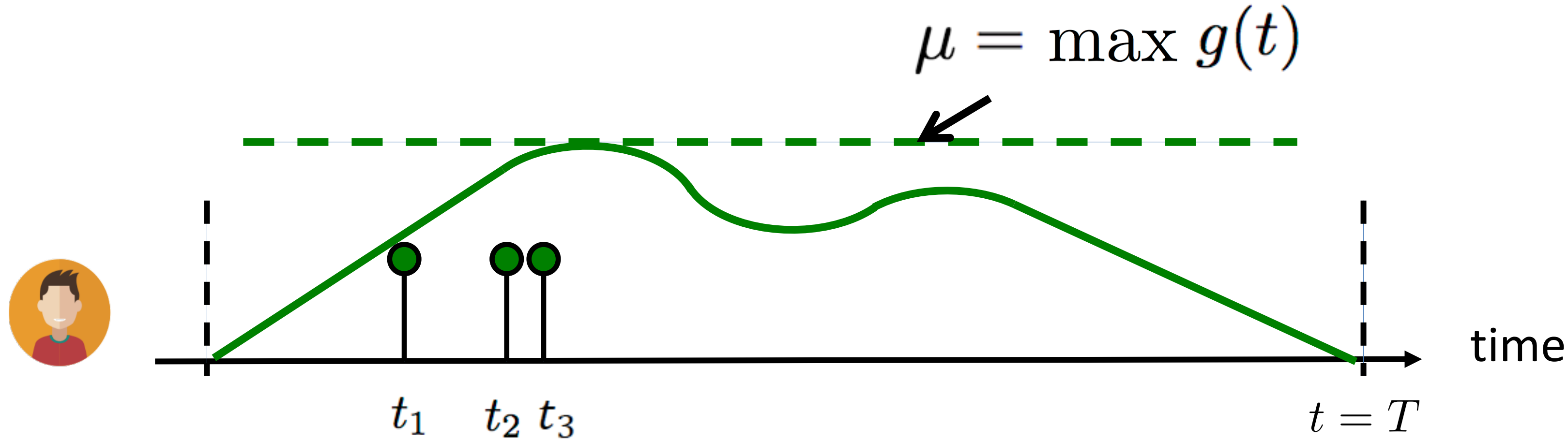
Positive combination of (Gaussian) RFB kernels:

$$\lambda^*(t) = \sum_j \alpha_j \overbrace{k(t - t_j)}$$





Sampling from an inhomogeneous Poisson



Thinning procedure (similar to rejection sampling):

1. **Sample** t **from Poisson process with intensity** μ

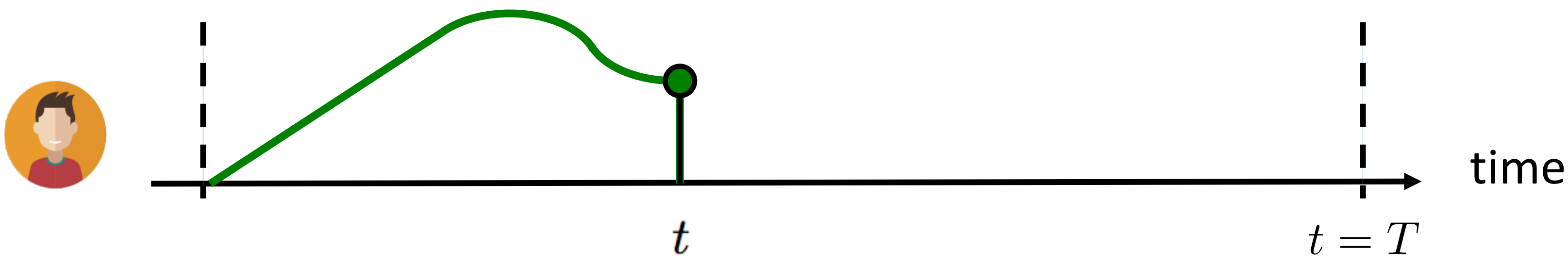
$$t \sim -\frac{1}{\mu} \log(1 - \underset{\uparrow}{u}) + t_3 \qquad u \sim Uniform(0,1)$$

Inversion sampling

2. **Generate** $u_2 \sim Uniform(0,1)$
3. **Keep the sample if** $u_2 \leq g(t) / \mu$

Keep sample with prob. $g(t)/\mu$

Terminating (or survival) process



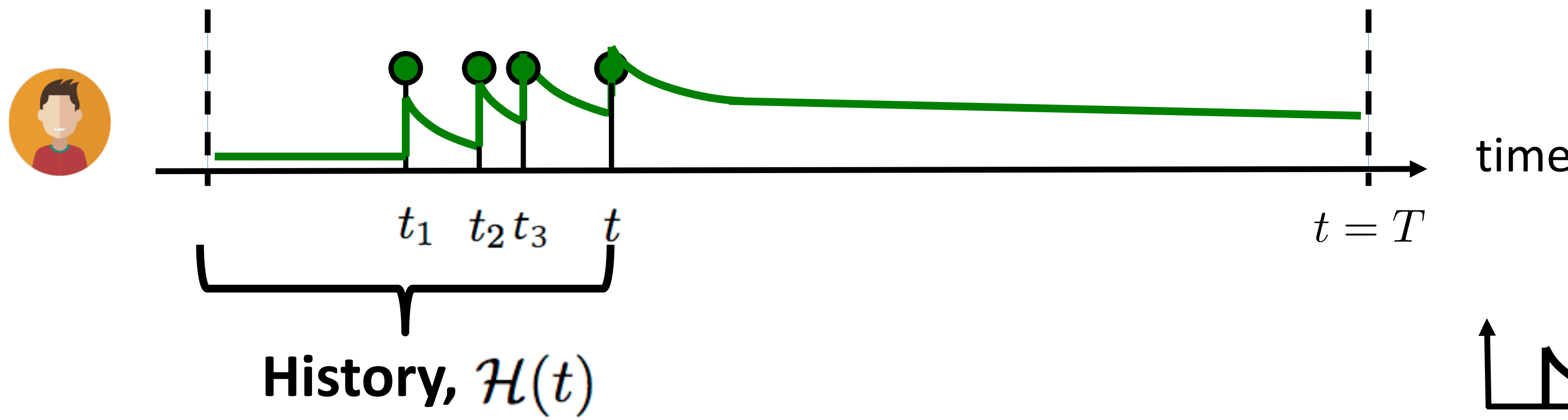
Intensity of a terminating (or survival) process

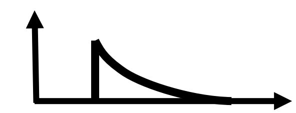
$$\lambda^*(t) = g^*(t)(1 - N(t)) \geqslant 0$$

Observations:

1. **Limited number of occurrences**

Self-exciting (or Hawkes) process





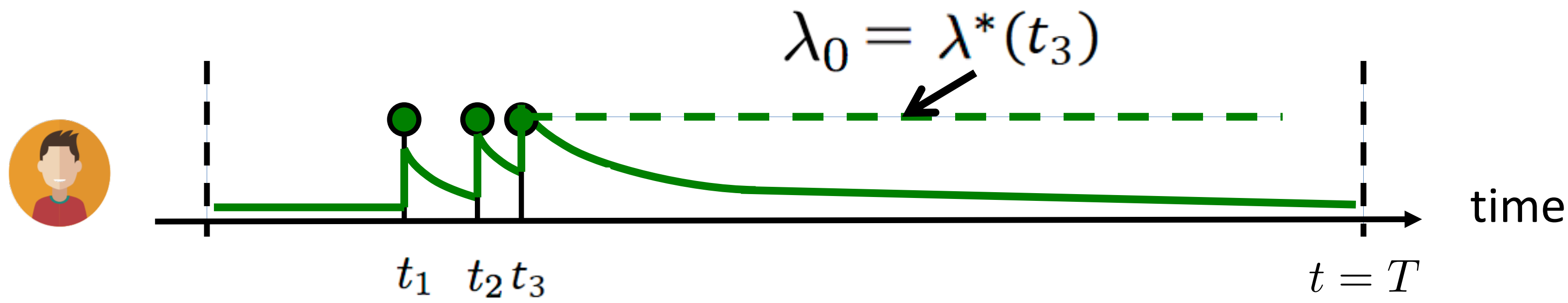
Intensity of self-exciting (or Hawkes) process:

$$\lambda^*(t) = \mu + \alpha \sum_{t_i \in \mathcal{H}(t)} \overbrace{\kappa_\omega(t - t_i)}^{\text{Triggering kernel}}$$
$$= \mu + \alpha \kappa_\omega(t) \star dN(t)$$

Observations:

1. **Clustered (or bursty) occurrence of events**
2. **Intensity is stochastic and history dependent**

Fitting a Hawkes process from a recorded timeline



$$\lambda^*(t_1)\lambda^*(t_2)\,\lambda^*(t_3)\cdots\cdots\lambda^*(t_n)\;\exp\left(-\int_0^T \lambda^*(\tau)\,d\tau\right)$$

$$\lambda^*(t) = \mu + \alpha \sum_{t_i \in \mathcal{H}(t)} \kappa_\omega(t - t_i)$$

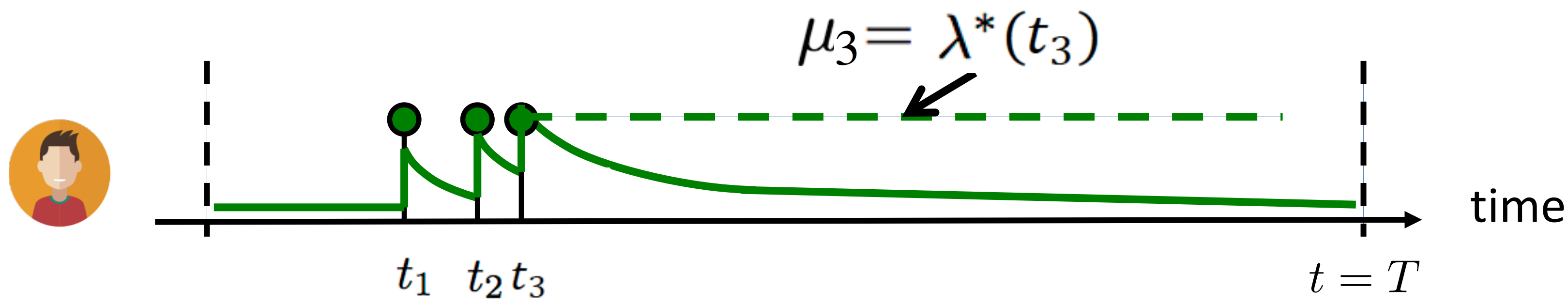
Maximum likelihood

$$\underset{\mu,\alpha}{\text{maximize}} \;\; \sum_{i=1}^{n} \log \lambda^*(t_i) - \int_0^T \lambda^*(\tau)\,d\tau$$

The max. likelihood is jointly convex in μ and α

(use CVX!)

Sampling from a Hawkes process



Thinning procedure (similar to rejection sampling):

1. **Sample** t **from Poisson process with intensity** μ_3

$$t \sim -\frac{1}{\mu_3} \log(1 - \underset{\uparrow \, Uniform(0,1)}{u}) + t_3$$

Inversion sampling

2. **Generate** $u_2 \sim Uniform(0, 1)$
3. **Keep the sample if** $u_2 \leq g(t) / \mu_3$

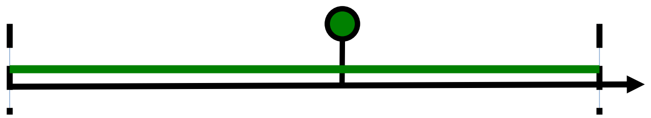
Keep sample with prob. $g(t)/\mu_3$

Summary

Building blocks to represent **different dynamic processes:**

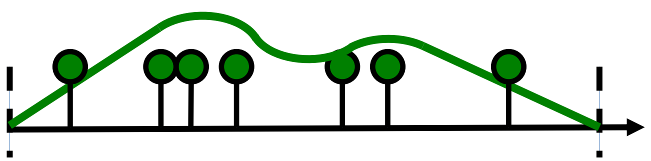
Poisson processes:

$$\lambda^*(t) = \lambda$$



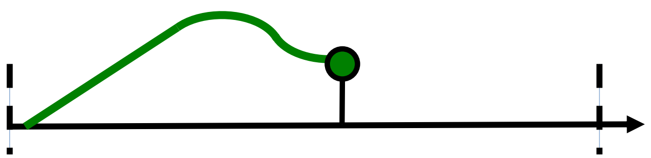
Inhomogeneous Poisson processes:

$$\lambda^*(t) = g(t)$$



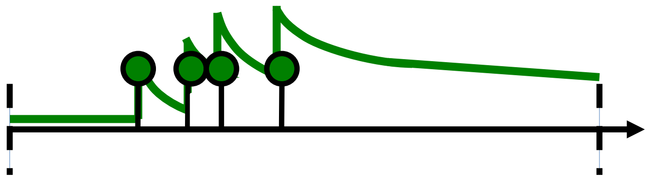
Terminating point processes:

$$\lambda^*(t) = g^*(t)(1 - N(t))$$



Self-exciting point processes:

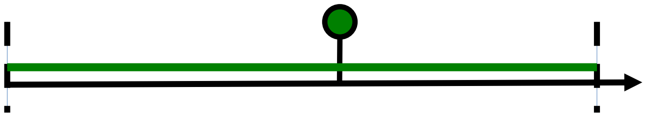
$$\lambda^*(t) = \mu + \alpha \sum_{t_i \in \mathcal{H}(t)} \kappa_\omega(t - t_i)$$



Summary

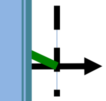
Building blocks to represent **different dynamic processes:**

Poisson processes:



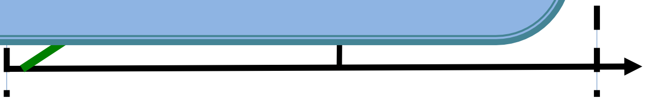
$$\lambda^*(t) = \lambda$$

Inho



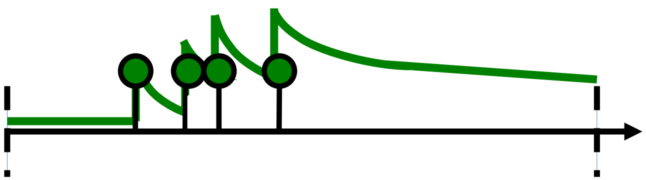
We know **how to fit** them
and **how to sample** from them

Term



$$\lambda^*(t) = g^*(t)(1 - N(t))$$

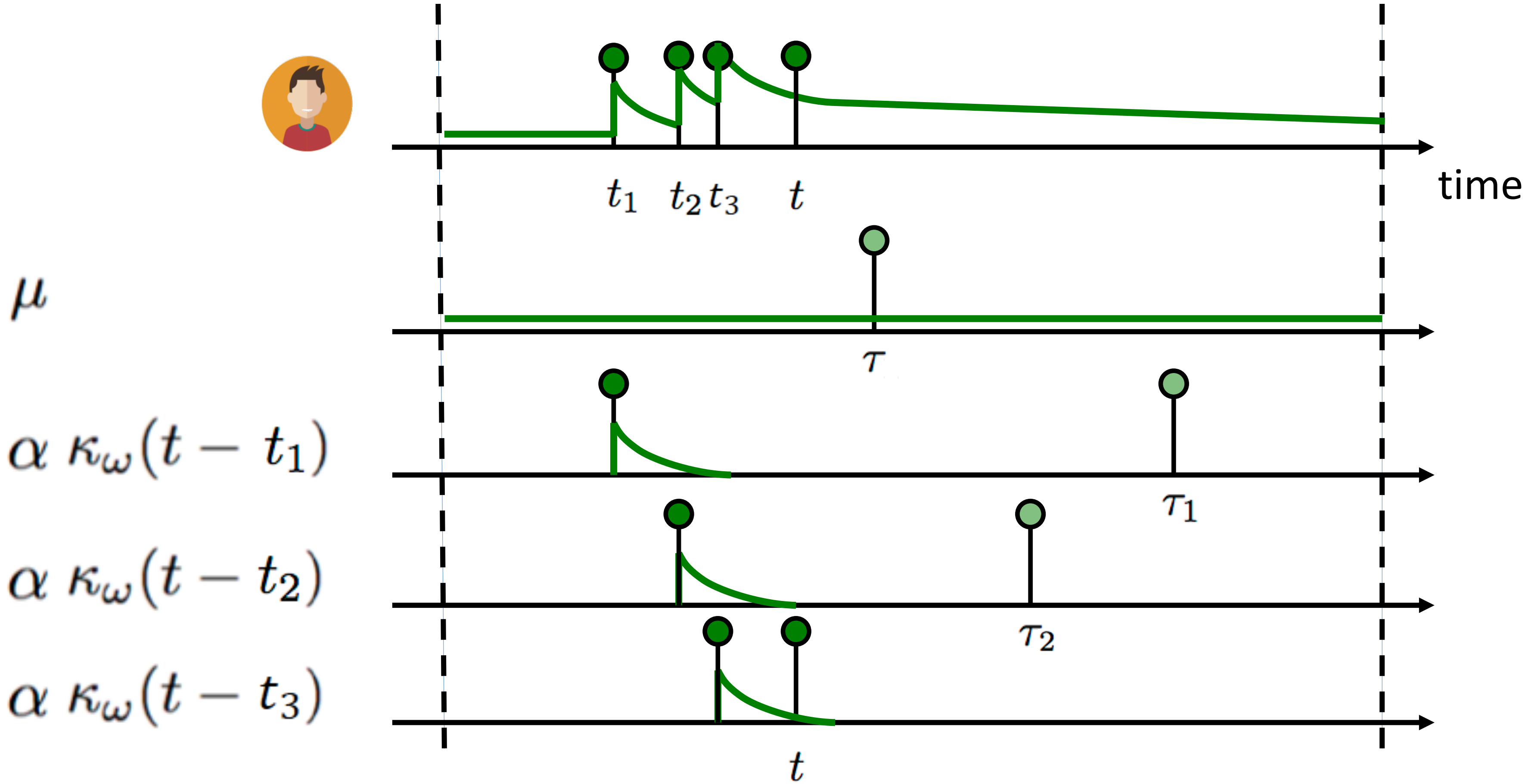
Self-exciting point processes:



$$\lambda^*(t) = \mu + \alpha \sum_{t_i \in \mathcal{H}(t)} \kappa_\omega(t - t_i)$$

Temporal Point Processes:
Superposition

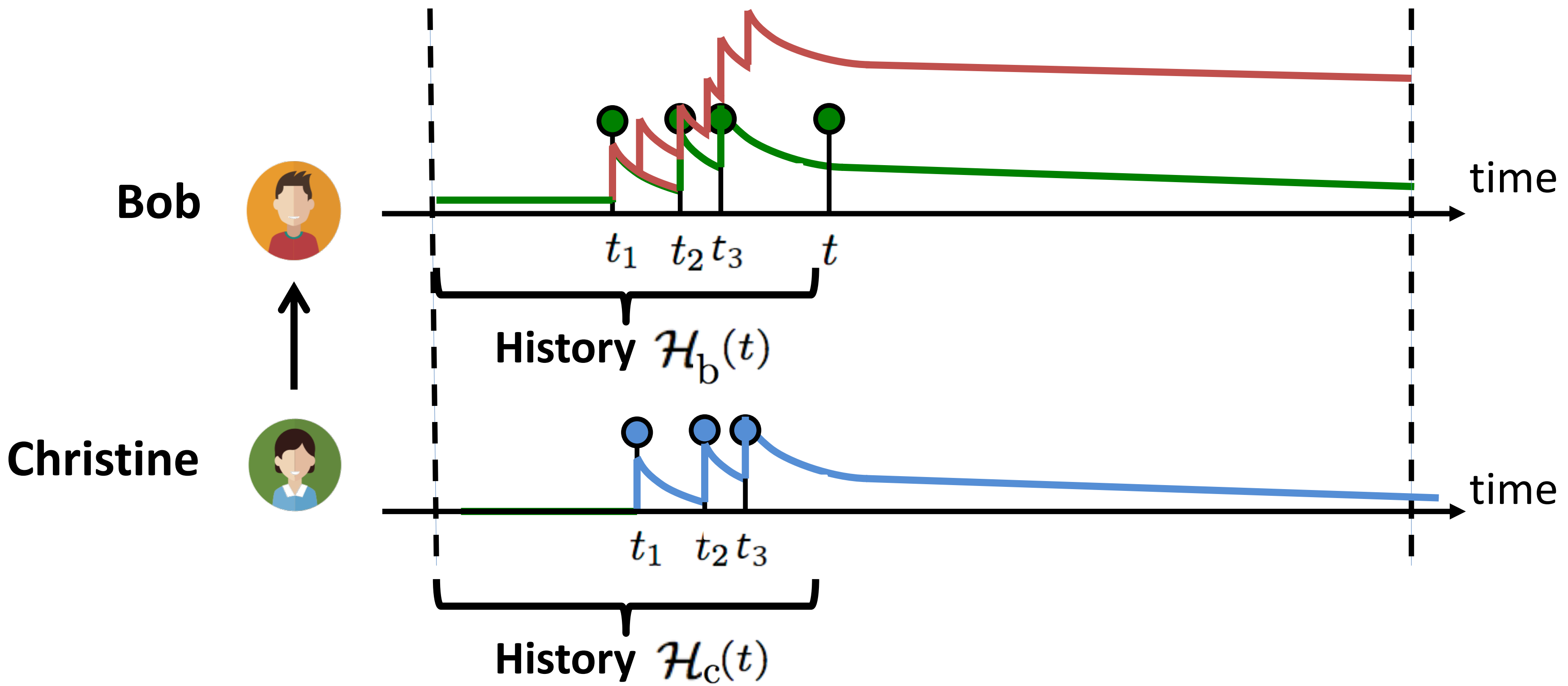
Superposition of processes



Sample each intensity + take minimum = Additive intensity

$$t = \min(\tau, \tau_1, \tau_2, \tau_3) \Rightarrow \lambda^*(t) = \mu + \alpha \sum_{t_i \in \mathcal{H}(t)} \kappa_\omega(t - t_i)$$

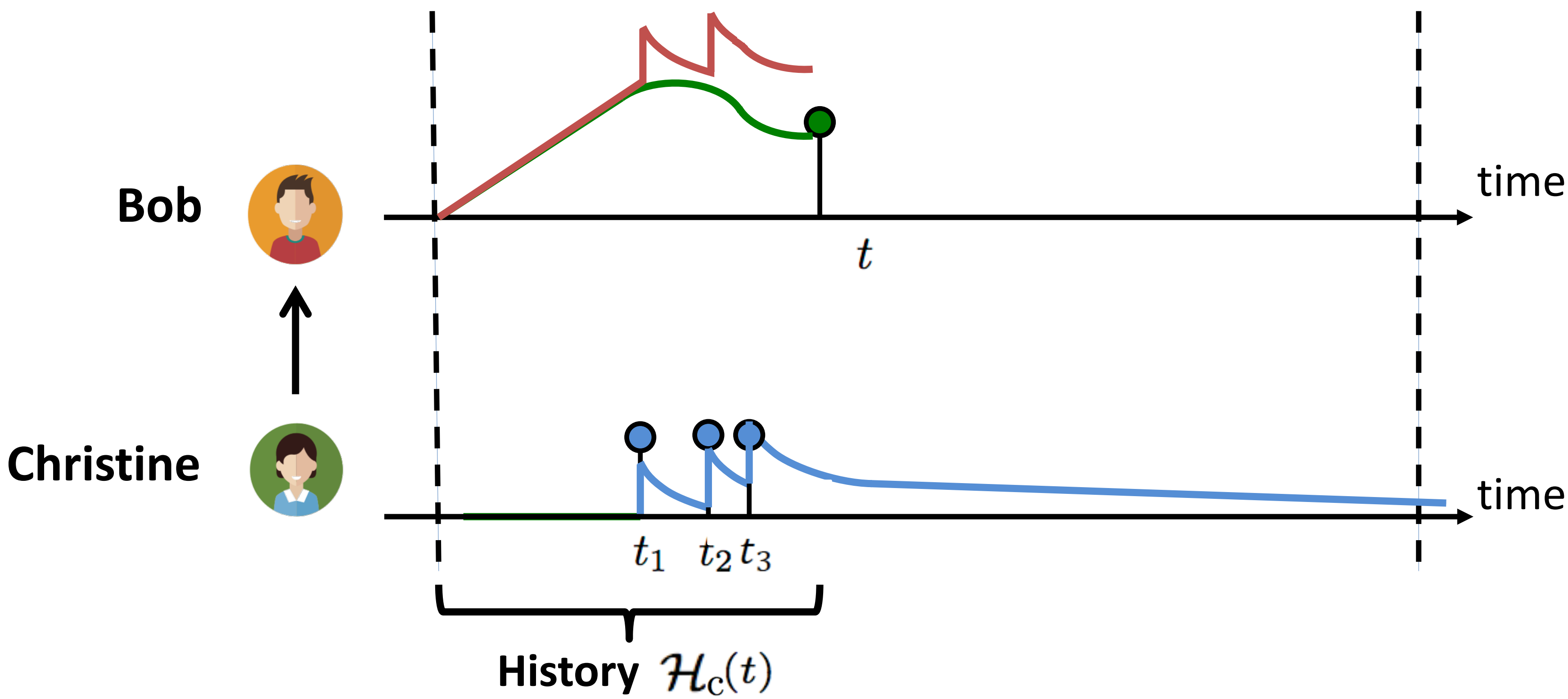
Mutually exciting process



Clustered occurrence affected by neighbors

$$\lambda^*(t) = \mu + \alpha \sum_{t_i \in \mathcal{H}_{\mathrm{b}}(t)} \kappa_\omega(t - t_i) + \beta \sum_{t_i \in \mathcal{H}_{\mathrm{c}}(t)} \kappa_\omega(t - t_i)$$

Mutually exciting terminating process



Clustered occurrence affected by neighbors

$$\lambda^*(t) = (1 - N(t))\Big(g(t) + \beta \sum_{t_i \in \mathcal{H}_c(t)} \kappa_\omega(t - t_i)\Big)$$